Exercise 93

A bacteria culture contains 200 cells initially and grows at a rate proportional to its size. After half an hour the population has increased to 360 cells.

(a) Find the number of bacteria after t hours.

(b) Find the number of bacteria after 4 hours.

(c) Find the rate of growth after 4 hours.

(d) When will the population reach 10,000?

Solution

Part (a)

Start with the assumption that the rate of population growth is proportional to the population.

$$\frac{dP}{dt} \propto P$$

Change this proportionality to an equation by introducing a constant.

$$\frac{dP}{dt} = kP$$

Divide both sides by P.

$$\frac{1}{P}\frac{dP}{dt} = k$$

Rewrite the left side as a derivative of a logarithm by using the chain rule.

$$\frac{d}{dt}\ln P = k$$

The function you take a derivative of to get k is $kt + C$, where C is any constant.

$$\ln P = kt + C$$

Exponentiate both sides to solve for P.

$$e^{\ln P} = e^{kt+C}$$

$$P(t) = e^C e^{kt}$$

Use a new constant P_0 for e^C.

$$P(t) = P_0 e^{kt}$$

Use the fact that the population is 200 initially to determine P_0.

$$P(0) = P_0 e^0 = 200 \quad \rightarrow \quad P_0 = 200$$

As a result,

$$P(t) = 200e^{kt}.$$

Use the fact that the population is 360 after 0.5 hours to determine k.

$$P(0.5) = 200e^{k(0.5)}$$

$$360 = 200e^{0.5k}$$

$$\frac{9}{5} = e^{0.5k}$$

$$\ln\frac{9}{5} = \ln e^{0.5k}$$

$$\ln\frac{9}{5} = (0.5k)\ln e$$

$$k = 2\ln\frac{9}{5}$$

Therefore, after t hours, the bacteria population is

$$\begin{aligned} P(t) &= 200e^{\left(2\ln\frac{9}{5}\right)t} \\ &= 200e^{\ln\left(\frac{9}{5}\right)^{2t}} \\ &= 200\left(\frac{9}{5}\right)^{2t}. \end{aligned}$$

Part (b)

Plug in $t = 4$ to get the bacteria population after 4 hours.

$$P(4) = 200\left(\frac{9}{5}\right)^{2(4)} \approx 22040 \text{ bacteria}$$

Part (c)

The rate of population growth after 4 hours is

$$\left.\frac{dP}{dt}\right|_{t=4} = kP(4) = \left(2\ln\frac{9}{5}\right)\left[200\left(\frac{9}{5}\right)^{2(4)}\right] \approx 25910 \text{ bacteria/hour.}$$

Part (d)

To find when the population will be 10,000, set $P(t) = 10\,000$ and solve the equation for t.

$$P(t) = 10\,000$$

$$200\left(\frac{9}{5}\right)^{2t} = 10\,000$$

$$\left(\frac{9}{5}\right)^{2t} = 50$$

$$\ln\left(\frac{9}{5}\right)^{2t} = \ln 50$$

$$2t\ln\left(\frac{9}{5}\right) = \ln 50$$

$$2t = \frac{\ln 50}{\ln\left(\frac{9}{5}\right)}$$

$$t = \frac{1}{2}\left(\frac{\ln 50}{\ln\frac{9}{5}}\right)$$

$$t \approx 3.33 \text{ hours}$$